

Proposta de resolução

Grupo I

1.

$$\overrightarrow{AM} \cdot \overrightarrow{AC} = (\overrightarrow{AD} + \overrightarrow{DM}) \cdot (\overrightarrow{AD} + \overrightarrow{DC}) = \overrightarrow{AD} \cdot \overrightarrow{AD} + \overrightarrow{AD} \cdot \overrightarrow{DC} + \overrightarrow{DM} \cdot \overrightarrow{AD} + \overrightarrow{DM} \cdot \overrightarrow{DC} =$$

$$= \overrightarrow{AD}^2 + \underbrace{0}_{\overrightarrow{AD} \perp \overrightarrow{DC}} + \underbrace{0}_{\overrightarrow{DM} \perp \overrightarrow{AC}} + \frac{1}{2} \overrightarrow{DC} \cdot \overrightarrow{DC} = \overrightarrow{AD}^2 + \overrightarrow{DC}^2 - \frac{1}{2} \overrightarrow{DC}^2 = \underbrace{\overrightarrow{AC}^2}_{\text{T. Pitágoras}} - \frac{1}{2} \overrightarrow{AB}^2$$

Opção (D)

2.

$$a = 4b \Leftrightarrow a = 2^{2b} \Leftrightarrow \log_2 a = 2b$$

$$\log_8 (32a^3) - \log_{32} (16a^2) = \frac{\log_2 (32a^3)}{\log_2 8} - \frac{\log_2 (16a^2)}{\log_2 32} = \frac{1}{3} (\log_2 32 + 3 \log_2 a) - \frac{1}{5} (\log_2 16 + 2 \log_2 a) =$$

$$= \frac{1}{3} (5 + 3 \times 2b) - \frac{1}{5} (4 + 2 \times 2b) = \frac{5}{3} + 2b - \frac{4}{5} - \frac{4}{5}b = \frac{13}{15} + \frac{6}{5}b$$

Opção (C)

3. Se a reta de equação $y = 3$ é uma reta tangente ao gráfico de f no ponto P , então $f'(a) = 0$.

$$\lim_{x \rightarrow a} \frac{f'(x)}{x - a} = 2 \Leftrightarrow \underbrace{\lim_{x \rightarrow a} \frac{f'(x) - f'(a)}{x - a}}_{\text{definição de derivada}} = 2 \Leftrightarrow f''(a) = 2$$

Como $f''(a) > 0$ e $f'(a) = 0$, então $f(a)$ é mínimo relativo de f .

Opção (C)

4.

$$f(1) = 3 - \log_2 (1+1) = 3 - 1 = 2$$

$$f\left(-\frac{1}{2}\right) = 3 - \log_2 \left(1 - \frac{1}{2}\right) = 3 - \log_2 \frac{1}{2} = 3 + 1 = 4 \quad P_3(1, 2)$$
$$f(7) = 3 - \log_2 (1+7) = 3 - 3 = 0 \quad P_4(7, 0)$$

$$P_1\left(-\frac{1}{2}, 4\right)$$

$$f(0) = 3 - \log_2 (1+0) = 3$$

$$P_2(0, 3)$$

$$f(15) = 3 - \log_2 (1+15) = 3 - 4 = -1$$

$$P_5(15, -1)$$

Para o segmento de reta intersetar a reta de equação $y = 1$ um dos extremos tem que ter ordenada maior que 1 e o outro extremo tem que ter ordenada menor que 1. Assim,

$$p = \frac{3 \times 2}{5C_2} = 0.6$$

Opção (C)

5.

$${}^{10}C_p \left(\frac{3}{\sqrt{x}} \right)^{10-p} \times \left(-\frac{x^2}{2} \right)^p = {}^{10}C_p \frac{3^{10-p} \times (-1)^p}{2^p} \times \frac{x^{2p}}{x^{5-\frac{p}{2}}} = {}^{10}C_p \frac{3^{10-p} \times (-1)^p}{2^p} \times x^{\frac{5}{2}p-5}$$

$$\frac{5}{2}p - 5 = 10 \Leftrightarrow p = 6$$

$$\therefore {}^{10}C_6 \frac{3^4 \times (-1)^6}{2^6} = \frac{8505}{32}$$

Opção (A)

6.

$${}^5C_2 \times 3! - 4 \times 3! = 36$$

- 5C_2 é o número de formas de escolher as duas posições para os livros iguais
- $3!$ é o número de trocas entre os livros diferentes
- 4 é o número de formas de colocar os livros iguais lado a lado

7.

$$m_r = \frac{1-0}{0+2} = \frac{1}{2}$$

$$f'(0) = m_r = \frac{1}{2}$$
$$f(0) = 1$$

$$r : y = \frac{1}{2}x + 1$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = m_r = \frac{1}{2}$$

$$\lim_{x \rightarrow +\infty} \left[x \left(f(x^{-1}) - 1 \right) + \frac{x}{2f(x)} \right] = \lim_{x \rightarrow +\infty} \frac{f\left(\frac{1}{x}\right) - 1}{\frac{1}{x}} + \frac{1}{2} \lim_{x \rightarrow +\infty} \frac{x}{f(x)} =$$
$$\underbrace{\lim_{y \rightarrow 0^+} \frac{f(y) - f(0)}{y-0}}_{\substack{\text{definição de derivada}}} + \frac{1}{2} \times \underbrace{\lim_{x \rightarrow +\infty} \frac{x}{f(x)}}_{\substack{\text{declive a.o.}}} = f'(0) + \frac{1}{2} \times \frac{1}{\frac{1}{2}} = \frac{1}{2} + 1 = \frac{3}{2}$$

Opção (C)

8.

$$A_{[OPQ]} = \frac{\overline{OP} \times \overline{RQ}}{2} = \frac{1 \times (-\tan \theta)}{2} = -\frac{\tan \theta}{2}$$

Opção (D)

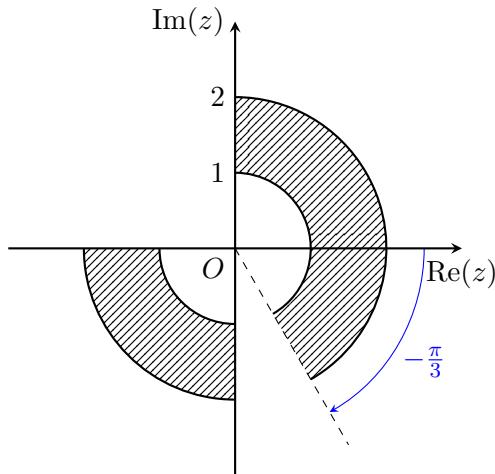
Grupo II

1.

1.1.

$$1 \leq |z| \leq 2 \wedge \left[\operatorname{Im}(z) \times \operatorname{Re}(z) \geq 0 \vee \left| \operatorname{Arg}(z) \right| < \frac{\pi}{3} \right] \Leftrightarrow$$

$$1 \leq |z| \leq 2 \wedge \left[(\operatorname{Im}(z) \geq 0 \wedge \operatorname{Re}(z) \geq 0) \vee (\operatorname{Im}(z) \leq 0 \wedge \operatorname{Re}(z) \leq 0) \vee -\frac{\pi}{3} < \operatorname{Arg}(z) < \frac{\pi}{3} \right]$$



$$A = \frac{\pi + \frac{\pi}{3}}{2} \times 2^2 - \frac{\pi + \frac{\pi}{3}}{2} \times 1^2 = \frac{8\pi}{3} - \frac{2\pi}{3} = 2\pi$$

1.2.

$$355 = 4 \times 88 + 3 \Rightarrow i^{355} = i^3 = -i$$

$$\left| \frac{1}{2} - \frac{\sqrt{3}}{2}i \right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\operatorname{Arg}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = -\arctan\sqrt{3} = -\frac{\pi}{3}$$

$$\begin{aligned} \frac{\left[(w_1)^7 - i^{355}\right] \times e^{i\frac{\pi}{3}}}{(\bar{w}_2)^3} &= \frac{\left[\left(e^{i\frac{\pi}{14}}\right)^7 + i\right] \times e^{i\frac{\pi}{3}}}{\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3} = \frac{\left(e^{i\frac{\pi}{2}+i}\right) \times e^{i\frac{\pi}{3}}}{\left(e^{i(-\frac{\pi}{3})}\right)^3} = \frac{2ie^{i\frac{\pi}{3}}}{e^{i(-\pi)}} = \frac{2e^{i\frac{\pi}{2}} \times e^{i\frac{\pi}{3}}}{e^{i(-\pi)}} = \\ &= 2e^{i(\frac{\pi}{2} + \frac{\pi}{3} + \pi)} = 2e^{i(\frac{11\pi}{6})} = 2e^{i(-\frac{\pi}{6})} \end{aligned}$$

Como o módulo do complexo é 2 e o seu argumento principal é $-\frac{\pi}{6}$ (note-se que $-\frac{\pi}{3} < -\frac{\pi}{6} < \frac{\pi}{3}$), então a imagem geométrica do complexo pertence à fronteira da região.

2.

$$P(\overline{A} \cup \overline{B}) = \frac{7}{9} \Leftrightarrow P(\overline{A \cap B}) = \frac{7}{9} \Leftrightarrow 1 - P(A \cap B) = \frac{7}{9} \Leftrightarrow P(A \cap B) = \frac{2}{9} \Leftrightarrow P(X = 3) = \frac{2}{9}$$

$$\begin{aligned} P(A|B) = \frac{1}{3} \Leftrightarrow \frac{P(A \cap B)}{P(B)} = \frac{1}{3} \Leftrightarrow \frac{\frac{2}{9}}{P(B)} = \frac{1}{3} \Leftrightarrow P(B) = \frac{2}{3} \Leftrightarrow P(X = 2) + P(X = 3) = \frac{2}{3} \Leftrightarrow \\ \Leftrightarrow P(X = 2) = \frac{2}{3} - \frac{2}{9} \Leftrightarrow P(X = 2) = \frac{4}{9} \end{aligned}$$

$$P(X = 1) + P(X = 2) + P(X = 3) = 1 \Leftrightarrow P(X = 1) = 1 - \frac{4}{9} - \frac{2}{9} \Leftrightarrow P(X = 1) = \frac{1}{3}$$

$X = x_i$	1	2	3
$P(X = x_i)$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{2}{9}$

3.

$$a_1 = \frac{1 \times f(1)}{2} = \frac{1 \times 2^2}{2} = 2$$

$$a_2 = \frac{1 \times f(2)}{2} = \frac{1 \times 2}{2} = 1$$

$$a_3 = \frac{1 \times f(3)}{2} = \frac{1 \times 2^0}{2} = \frac{1}{2}$$

...

(a_n) é uma progressão geométrica de razão $\frac{1}{2}$.

Note-se que:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1 \times f(n+1)}{2}}{\frac{1 \times f(n)}{2}} = \frac{2^{3-n-1}}{2^{3-n}} = 2^{3-n-1-3+n} = \frac{1}{2} = r$$

A soma das áreas de todos os triângulos é a soma de todos os termos de uma progressão geométrica (de razão r , com $|r| < 1$), que é dada por:

$$S = \frac{a_1}{1-r} = \frac{2}{1-\frac{1}{2}} = 4$$

4.

4.1.

$$f(0) = 3 - 3(0+1)e^{2 \times 0} = 3 - 3 = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} [3 - 3(x+1)e^{2x}] = 3 - 3(0+1)e^{2 \times 0} = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x \ln x) \underset{y = \frac{1}{x}}{\underset{y \rightarrow +\infty}{=}} \lim_{y \rightarrow +\infty} \frac{1}{y} \ln \left(\frac{1}{y} \right) = \lim_{y \rightarrow +\infty} \frac{\ln(y^{-1})}{y} = -\underbrace{\lim_{y \rightarrow +\infty} \frac{\ln y}{y}}_{\text{limite notável}} = -0 = 0$$

Como $f(0) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$, então f é contínua em $x = 0$.

4.2.

$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} [3 - 3(x+1)e^{2x}] = \lim_{x \rightarrow -\infty} 3 - 3 \lim_{x \rightarrow -\infty} (x+1)e^{2x} = \\ &\underset{y = -2x}{\underset{y \rightarrow +\infty}{=}} 3 - 3 \lim_{y \rightarrow +\infty} \left(-\frac{1}{2}y + 1 \right) e^{-y} = 3 - 3 \lim_{y \rightarrow +\infty} \frac{-\frac{1}{2}y + 1}{e^y} = 3 - 3 \times \frac{\lim_{y \rightarrow +\infty} \left(-\frac{1}{2} + \frac{1}{y} \right)}{\lim_{y \rightarrow +\infty} \frac{e^y}{y}} = \\ &= 3 - 3 \times \frac{-\frac{1}{2}}{+\infty} = 3 - 3 \times 0 = 3 \end{aligned}$$

A reta de equação $y = 3$ é assíntota horizontal ao gráfico da função f quando $x \rightarrow -\infty$.

$$m = \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{x \ln x}{x} = \lim_{x \rightarrow +\infty} \ln x = \ln(+\infty) = +\infty$$

O gráfico de f não admite assíntotas não verticais quando $x \rightarrow +\infty$.

4.3.

$$A\left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right) \Leftrightarrow A\left(-\frac{3}{2}, 3 + \frac{3}{2e^3}\right) \Leftrightarrow A\left(-\frac{3}{2}, \frac{6e^3 + 3}{2e^3}\right)$$

$$B(b, f(b)) \Leftrightarrow B(b, b \ln b)$$

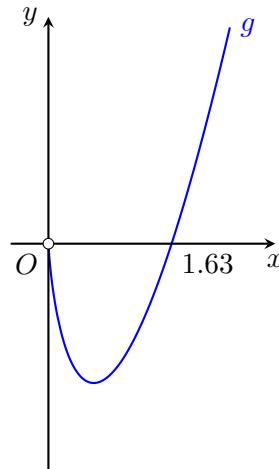
$$\overrightarrow{OA} = A - O = \left(-\frac{3}{2}, \frac{6e^3 + 3}{2e^3}\right)$$

$$\overrightarrow{OB} = B - O = (b, b \ln b)$$

$\angle AOB$ é obtuso se e só se:

$$\overrightarrow{OA} \cdot \overrightarrow{OB} < 0 \Leftrightarrow \left(-\frac{3}{2}, \frac{6e^3 + 3}{2e^3}\right) \cdot (b, b \ln b) < 0 \Leftrightarrow -\frac{3}{2}b + \frac{6e^3 + 3}{2e^3} \times b \ln b < 0$$

Seja g a função definida por $g(x) = -\frac{3}{2}x + \frac{6e^3 + 3}{2e^3}x \ln x$



$\therefore \angle AOB$ é obtuso se $b \in]0, 1.63[$

5.

5.1. $MV \perp ABC$, portanto vamos determinar um vetor normal ao plano ABC , que será um vetor diretor da reta MV .

$$\begin{aligned} \overrightarrow{AB} &= \sqrt{0^2 + 4^2 + 3^2} = 5 = \overrightarrow{AD} \\ A(0, 0, 3) & & \overrightarrow{AD} &= (5, 0, 0) \\ B(0, 4, 0) & & \overrightarrow{AB} &= B - A = (0, 4, -3) \end{aligned}$$

Seja $\vec{n} = (a, b, c)$ um vetor normal ao plano ABC . Logo,

$$\begin{cases} \vec{n} \cdot \overrightarrow{AD} = 0 \\ \vec{n} \cdot \overrightarrow{AB} = 0 \end{cases} \Leftrightarrow \begin{cases} 5a = 0 \\ 4b - 3c = 0 \end{cases} \Leftrightarrow \begin{cases} a = 0 \\ b = \frac{3}{4}c \end{cases}$$

Fazendo $c = 4$, por exemplo, obtém-se $\vec{n} = (0, 3, 4)$

$$\vec{n} \parallel MV$$

$C = (5, 4, 0)$

M é o ponto médio de $[AC]$

$$M\left(\frac{0+5}{2}, \frac{0+4}{2}, \frac{3+0}{2}\right) \Leftrightarrow M\left(\frac{5}{2}, 2, \frac{3}{2}\right) \in MV$$

$$\therefore MV : (x, y, z) = \left(\frac{5}{2}, 2, \frac{3}{2}\right) + k(0, 3, 4), \quad k \in \mathbb{R}$$

5.2.

$$V = 25 \Leftrightarrow 25 = \frac{1}{3}A_b \times h \Leftrightarrow 25 = \frac{1}{3} \times \frac{5^2}{2} \times \overrightarrow{MV} \Leftrightarrow \overrightarrow{MV} = 6$$

$$V = M + \overrightarrow{MV}$$

Vamos determinar \overrightarrow{MV} :

$$\overrightarrow{MV} \parallel \vec{n} = (0, 3, 4)$$

$$\|\vec{n}\| = \sqrt{0^2 + 3^2 + 4^2} = 5$$

Assim,

$$\overrightarrow{MV} = \frac{6}{5}\vec{n} \vee \overrightarrow{MV} = -\frac{6}{5}\vec{n} \Leftrightarrow \overrightarrow{MV} = \left(0, \frac{18}{5}, \frac{24}{5}\right) \vee \underbrace{\overrightarrow{MV} = \left(0, -\frac{18}{5}, -\frac{24}{5}\right)}_{\text{tem sentido contrário}} \Rightarrow \overrightarrow{MV} = \left(0, \frac{18}{5}, \frac{24}{5}\right)$$

$$\therefore V = M + \overrightarrow{MV} = \left(\frac{5}{2}, 2, \frac{3}{2}\right) + \left(0, \frac{18}{5}, \frac{24}{5}\right) = \left(\frac{5}{2}, \frac{28}{5}, \frac{63}{10}\right)$$

6.

6.1.

$$g'(x) = [4(\sin(2x) + x)]' = 4(\sin(2x) + x)' = 4(2\cos(2x) + 1) = 4[2(\cos^2 x - \sin^2 x) + 1] =$$

$$= 4[2(1 - \sin^2 x - \sin^2 x) + 1] = 4(2 - 4\sin^2 x + 1) = 4(3 - 4\sin^2 x)$$

6.2.

$$g\left(\frac{\pi}{12}\right) = 4\left[\sin\left(2 \times \frac{\pi}{12}\right) + \frac{\pi}{12}\right] = 4\left(1 + \frac{\pi}{12}\right) = 2 + \frac{\pi}{3} = \frac{6 + \pi}{3}$$

$$A\left(\frac{\pi}{12}, \frac{6 + \pi}{3}\right)$$

$$g'(x) = 0 \Leftrightarrow 4(3 - 4\sin^2 x) = 0 \Leftrightarrow \sin^2 x = \frac{3}{4} \Leftrightarrow \sin x = \pm \frac{\sqrt{3}}{2} \Leftrightarrow \sin x = \frac{\sqrt{3}}{2} \vee \underbrace{\sin x = -\frac{\sqrt{3}}{2}}_{\text{eq. imp. em }]0, \pi[}$$

$$\Rightarrow x = \frac{\pi}{3} + 2k\pi \vee x = \frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

As únicas soluções em $]0, \pi[$ são: $x = \frac{\pi}{3} \vee x = \frac{2\pi}{3}$

x	0		$\frac{\pi}{3}$		$\frac{2\pi}{3}$		π
$g'(x)$			+	0	-	0	+
g			↗	Max.	↘	Min.	↗

$$g\left(\frac{\pi}{3}\right) = 4 \left[\sin\left(\frac{2\pi}{3}\right) + \frac{\pi}{3} \right] = 4 \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) = \frac{6\sqrt{3} + 4\pi}{3}$$

$$B\left(\frac{\pi}{3}, \frac{6\sqrt{3} + 4\pi}{3}\right)$$

$$D\left(\frac{2\pi}{3}, \frac{6\sqrt{3} + 4\pi}{3}\right)$$

$$\overline{BD} = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$$

$$h = y_B - y_A = \frac{6\sqrt{3} + 4\pi}{3} - \frac{6 + \pi}{3} = 2\sqrt{3} - 2 + \pi = \pi + 2(\sqrt{3} - 1)$$

$$\therefore A_{[ABD]} = \frac{\overline{BD} \times h}{2} = \frac{\frac{\pi}{3} \times (\pi + 2(\sqrt{3} - 1))}{2} = \frac{\pi}{6} [\pi + 2(\sqrt{3} - 1)]$$

6.3.

$$g''(x) = \left[4(3 - 4\sin^2 x) \right]' = 4(3 - 4\sin^2 x)' = 4(0 - 8\sin x \cos x) = -16\sin(2x)$$

$$g''(x) = 0 \Leftrightarrow -16\sin(2x) = 0 \Leftrightarrow \sin(2x) = 0 \Leftrightarrow 2x = k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{k\pi}{2}, k \in \mathbb{Z}$$

A única solução em $]0, \pi[$ é $x = \frac{\pi}{2}$.

x	0		$\frac{\pi}{2}$		π
$g''(x)$	-	-	0	+	+
g	↑	↙	P.I.	↙	↑

$$g\left(\frac{\pi}{2}\right) = 4 \left[\sin\left(2 \times \frac{\pi}{2}\right) + \frac{\pi}{2} \right] = 2\pi$$

$$P\left(\frac{\pi}{2}, 2\pi\right)$$

$$m_r = g'\left(\frac{\pi}{2}\right) = 4 \left(3 - 4\sin^2 \frac{\pi}{2} \right) = -4$$

$$\therefore y - 2\pi = -1 \left(x - \frac{\pi}{2} \right) \Leftrightarrow y - 2\pi = -4x + 2\pi \Leftrightarrow 4x + y = 4\pi$$